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Behavior of Partial Sums of Wavelet Series

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Given a distribution f belonging the Sobolev space $H^{1/2}$, we show that partial sums of its wavelet expansion behave like truncated versions of the inverse Fourier transform of $\hat{f}$. Our result is sharp in the sense that such behavior no longer happens in general for H^s if $s<1/2$. © 2000 Academic Press

1. INTRODUCTION

Suppose ψ is a smooth function with "good" oscillation and rapid decay at infinity and such that the family of functions

$$\psi_{j,k}(x)=2^{j/2}\psi(2^jx-k), \qquad j,k\in Z$$

is an orthonormal basis for $L^2(R)$. Such a function ψ is commonly called a mother wavelet and the functions $\psi_{j,k}$ are called wavelets.

Pointwise convergence of wavelet expansions have been studied extensively in [3, 4] and [8]. In [4], it is shown that the wavelet expansions of a function belonging to L^p converge pointwise everywhere on the Lebesgue set of the given function, for $1\leqslant p<\infty$. In [8], precise convergence rates are obtained for wavelet expansions of continuous functions in a given Sobolev space.

This note complements the above-mentioned papers which do not discuss behavior at points outside the Lebesgue set of the function in question. Our main result describes behavior of wavelet expansions at each point of the real line, and thus may be used to determine behavior at singularities.

Our goal in this note is to identify classes of functions f such that

$$S_nf(x)-\frac{1}{\sqrt{2\pi}}\int_{I_n}\hat{f}(\omega)\,e^{i\omega x}\,d\omega\to 0, \qquad n\to\infty \tag{1}$$

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uniformly on the real line, where

$$S_n f(x) = \sum_{j=-n}^{n} \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k} \rangle \, \psi_{j,k}(x), \tag{2}$$

and $I_n = \{t \in \mathbf{R} : 2^{-n} < \alpha^{-1}|t| < 2^n\}$ where α is some positive constant independent of f. Here,

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \, \overline{g(x)} \, dx.$$

Along the scale of Sobolev spaces H^s, $-\infty < s < \infty$, we shall prove that the smallest real number s for which (1) holds uniformly on the real line, is $s = 1/2$. Given a real number s, recall that H^s is the space of all tempered distributions f such that the Fourier transform $\hat{f}$ belongs to L^2_{loc} and

$$\int_{-\infty}^{\infty} (1 + |\omega|^2)^s \, |\hat{f}(\omega)|^2 \, d\omega < \infty.$$

We define the Fourier transform of f by

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, e^{-i\omega x} \, dx.$$

Hence, the inversion formula becomes

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) \, e^{i\omega x} \, d\omega.$$

2. AN IDENTITY FOR THE PARTIAL SUMS

Throughout this paper, ψ will denote Yves Meyer's Littlewood–Paley mother wavelet, see [6, p. 74]. Hence, ψ is a function from the Schwartz class $\mathscr{S}(R)$ and its Fourier transform is supported on $I \cup 2I$, with $I = [-2a, -a] \cup [a, 2a]$ for some $a > 0$. See also [1] for a description of band-limited wavelets. $\mathscr{S}'(R)$ shall denote the space of tempered distributions. If $f \in \mathscr{S}'(R)$ and $\phi \in \mathscr{S}(R)$, we write $\langle f, \phi \rangle = f(\overline{\phi})$. Z will denote the Zak transform of ψ:

$$Z(x, \xi) = \sum_{k=-\infty}^{\infty} \psi(x-k) \, e^{i\xi k}$$

THEOREM 1. *Let* $f \in \mathscr{S}'(R)$ *such that* $\hat{f} \in L^1_{loc}$. *Given a real number* x *and a positive integer* n,

$$S_n f(x) = \frac{1}{\sqrt{2\pi}} \int_{I_n} \hat{f}(\omega)\, e^{i\omega x}\, d\omega + R_n f(x) + r_n f(x), \tag{3}$$

where $S_n f(x)$ *has been defined in* (2), $I_n = \bigcup_{s=-n+1}^{n} 2^s I$ *and*

$$r_n f(x) = \int_{2^{-n} I} \hat{f}(\omega)\, \overline{\hat{\psi}(2^n \omega)}\, Z(2^{-n} x, 2^n \omega)\, d\omega, \tag{4}$$

$$R_n f(x) = \int_{2^{n+1} I} \hat{f}(\omega)\, \overline{\hat{\psi}(2^{-n} \omega)}\, Z(2^n x, 2^{-n} \omega)\, d\omega. \tag{5}$$

The proof of Theorem 1 is broken down into two lemmas.

LEMMA 1. *Let* f *be as in Theorem* 1. *Given a real number* x *and a positive integer* n,

$$S_n f(x) = \sum_{s=-n+1}^{n} \int_{2^s I} \hat{f}(\omega)\, H_s(2^{-s}\omega, x)\, d\omega + R_n f(x) + r_n f(x)$$

where

$$H_s(\xi, x) = \overline{\hat{\psi}(\xi)}\, Z(2^s x, \xi) + \overline{\hat{\psi}(2\xi)}\, Z(2^{s-1} x, 2\xi).$$

Proof of Lemma 1. Applying Parseval's identity, we may write

$$S_n f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega)\, K_n(\omega, x)\, d\omega$$

where

$$K_n(\omega, x) = \sum_{j=-n}^{n} \overline{\hat{\psi}(2^{-j}\omega)}\, Z(2^j x, 2^{-j}\omega).$$

Since the support of $K_n(\cdot, x)$ is contained in $\bigcup_{l=-n}^{n+1} 2^l I$, we have

$$S_n f(x) = \sum_{s=-n}^{n+1} \int_{2^s I} \hat{f}(\omega)\, K_n(\omega, x)\, d\omega$$

Now, we isolate the terms in the summation corresponding to $s = -n$ and $s = n+1$. Observe that $\hat{\psi}(2^{-j}\omega) = 0$ if $\omega \in 2^{-n} I$ and $-n+1 \leqslant j \leqslant n$. Hence,

$$K_n(\omega, x) = \overline{\hat{\psi}(2^n \omega)}\, Z(2^{-n} x, 2^n \omega), \qquad \omega \in 2^{-n} I. \tag{6}$$

Similarly, we obtain

$$K_n(\omega, x) = \overline{\hat{\psi}(2^{-n}\omega)}\, Z(2^n x, 2^{-n}\omega), \qquad w \in 2^{n+1}I. \tag{7}$$

The last two equations above yield the desired forms of $r_n f(x)$ and $R_n f(x)$ respectively.

If $-n+1 \leqslant s \leqslant n$, then $\hat{\psi}(2^{-j}\omega) = 0$ whenever $\omega \in 2^s I$ and either $s+1 \leqslant j \leqslant n$ or $-n \leqslant j \leqslant s-2$. Hence,

$$K_n(\omega, x) = \overline{\hat{\psi}(2^{1-s}\omega)}\, Z(2^{s-1}x, 2^{1-s}\omega) + \overline{\hat{\psi}(2^{-s}\omega)}\, Z(2^s x, 2^{-s}\omega)$$

if $\omega \in 2^s I$ with $-n+1 \leqslant s \leqslant n$. This completes the proof of Lemma 1.

QED.

In the next lemma, we consider the wavelet expansion of $\exp(i2^s\xi x)$ where $\xi \in I$ and s is an integer.

Lemma 2. *Given $x \in R$, $\xi \in I$, $s \in Z$,*

$$\frac{\exp(i2^s\xi x)}{\sqrt{2\pi}} = H_s(\xi, x). \tag{8}$$

where $H_s(\xi, x)$ has been defined in the statement of Lemma 1.

Proof of Lemma 2. Fix a real number x_0, an integer k, and define $f \in L^2(R)$ such that

$$\hat{f}(\omega) = \overline{H_k(2^{-k}\omega, x_0)} - \frac{\exp(-i\omega x_0)}{\sqrt{2\pi}}, \qquad \text{if} \quad \omega \in 2^k I,$$

and $\hat{f}(\omega) = 0$, otherwise.

If $-n < k < n+1$, then by Lemma 1,

$$S_n f(x) = 2^k \int_I \left(\overline{H_k(\xi, x_0)} - \frac{\exp(-i2^k\xi x_0)}{\sqrt{2\pi}} \right) H_k(\xi, x)\, d\xi.$$

Observe that $S_n f$ is independent of n. Since $S_n f$ converges to f in $L^2(R)$, then for almost every x,

$$\int_I \left(\overline{H_k(\xi, x_0)} - \frac{\exp(-i2^k\xi x_0)}{\sqrt{2\pi}} \right) H_k(\xi, x)\, d\xi$$
$$= \frac{1}{\sqrt{2\pi}} \int_I \left(\overline{H_k(\xi, x_0)} - \frac{\exp(-i2^k\xi x_0)}{\sqrt{2\pi}} \right) \exp(i2^k\xi x)\, d\xi.$$

Since both sides are continuous functions of x, equality must hold everywhere. For $x = x_0$, the equality may be written

$$\int_I \left| H_k(\xi, x_0) - \frac{\exp(i2^k \xi x_0)}{\sqrt{2\pi}} \right|^2 d\xi = 0.$$

This completes the proof of Lemma 2.

3. WAVELET EXPANSIONS IN $H^{1/2}$

Our main results are applications of the formulae obtained in the preceding section.

THEOREM 2. *If $f \in H^{1/2}$, then*

$$\sup_{x \in R} \left| S_n f(x) - \frac{1}{\sqrt{2\pi}} \int_{I_n} \hat{f}(\omega)\, e^{i\omega x} d\omega \right| \to 0, \qquad as \qquad n \to \infty. \tag{9}$$

This result is sharp as the following example shows. Recall that $I_n = \bigcup_{s=-n+1}^{n} 2^s I$, the support of $\hat{\psi}$ is contained in $I \cup 2I$ and $I = [-2a, -a] \cup [a, 2a]$ where $a > 0$.

Fix a real number x_0 and define $f \in L^2(R)$ such that $\hat{f}(\omega) = 0$ if $|\omega| \leqslant 2a$ and

$$\hat{f}(\omega) = 2^{-n-1} \hat{\psi}(2^{-n}\omega)\, \overline{Z(2^n x_0, 2^{-n}\omega)} \qquad \text{if} \quad \omega \in 2^{n+1} I,$$

for $n = 0, 1, 2, 3, \ldots$. The boundedness of Z shows that for each $s < 1/2$, $f \in H^s$. Indeed,

$$\int_{-\infty}^{\infty} |w|^{2s}\, |\hat{f}(\omega)|^2\, d\omega \leqslant C \sum_{n=0}^{\infty} 2^{n(2s-1)} \int_{2I} |\xi|^{2s}\, |\hat{\psi}(\xi)|^2\, d\xi$$

where $4C = \sup\{|Z(x, \xi)|^2 : x, \xi \in R\}$.

However, (9) does not hold. If it were true that

$$S_n f(x_0) - \frac{1}{\sqrt{2\pi}} \int_{I_n} \hat{f}(\omega)\, e^{i\omega x_0}\, d\omega \to 0,$$

then by Theorem 1, $r_n f(x_0) + R_n f(x_0) \to 0$. Observe that

$$R_n f(x_0) = \frac{1}{2} \int_{2I} |\hat{\psi}(\omega)\, Z(2^n x_0, \omega)|^2\, d\omega$$

while $r_n f(x_0) = 0$ for each positive integer n.

This would imply that for some sequence of positive integers $\{n_k\}$,

$$\lim_{k \to \infty} |\hat{\psi}(\omega)\, Z(2^{n_k} x_0, \omega)| = 0$$

for almost every ω in $2I$. In view of Lemma 2, this would imply that

$$\lim_{k \to \infty} |\hat{\psi}(\omega)\, Z(2 \cdot 2^{n_k} x_0, \omega)| = \frac{1}{\sqrt{2\pi}}$$

for almost every ω in I. This is impossible since Z is bounded while $\hat{\psi}$ is continous with $\hat{\psi}(a) = 0$.

Proof of Theorem 2. We only have to show that both $R_n f$ and $r_n f$ tend to zero uniformly on R. Using Hölder's inequality and the boundedness of Z, we obtain for all real numbers x,

$$|R_n f(x)|^2 \leqslant b_n \int_{2^{n+1} I} |\omega| \cdot |\hat{f}(\omega)|^2 \, d\omega \qquad \text{where} \qquad b_n \leqslant C \int_{2I} \frac{d\xi}{|\xi|},$$

and the constant C depends only on the mother wavelet ψ. Hence if f belongs to $H^{1/2}$,

$$\sup_{x \in R} |R_n f(x)| \to 0,$$

as n tends to ∞. Similarly, we have $\sup_{x \in R} |r_n f(x)| \to 0$.

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